Common Distributions

~~Functions of random variables, Expectation~~

~~Putting a value on random variables~~

How we model the real world of randomness

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Review: functions of random variables

$X = (\Omega, \mathbb{P})$

$x \in \Omega$

e.g. 3 indep. Bernoulli r.v. each w/p p:

$\Omega = \{0,1\} \times \{0,1\} \times \{0,1\}$ (e.g. $(0,1,0) \in \Omega$)

$\underline{\mathbb{P}: \; \mathbb{P}(e_1)\mathbb{P}(e_2)\mathbb{P}(e_3)}$

$f(x) = x_1 + x_2 + x_3$

$\Omega' = \{0,1,2,3\}$

$f(X) := \boxed{(\Omega', \mathbb{P}')}$

$\mathbb{P}'$, $\mathbb{P}'(f(X)=1) := \mathbb{P}(\text{exactly 1 heads})$

$= \left| \sum_{x : f(x) = 1} \mathbb{P}(x) \right| = 3p(1-p)^2$

$(1,0,0)$, $(0,1,0)$, $(0,0,1)$
p $(1-p)(1-p)$

Review: Expectation

Definition. *The expectation of a numeric random variable $X = (\Omega, \mathbb{P})$ is given by the following:*

$$\mathrm{E}[X] := \sum_{x \in \Omega} x \mathbb{P}(X = x).$$

E.g. Bernoulli, $p = 0.5$, $\mathrm{E}[X] = 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5$

Expectation of (functions of) random variables

$$E[f(X)] = \sum_{i \in \Omega'} i\, \mathbb{P}(f(x) = i)$$

$$= \sum_{i \in \Omega'} \left(\sum_{x \mid f(x) = i} i\, \mathbb{P}(x) \right) \quad i = f(x)$$

$$= \boxed{\sum_{x \in \Omega} f(x)\, \mathbb{P}(x)}$$

(double sum reindex, see expectation equality proof)

e.g. $E(X^2) = \sum x^2\, \mathbb{P}(x)$ "$f(x) = x^2$"

$(E(X+Y))$

$E(XY)$

$E(X^2)$

$f: (X \times Y) \to \mathbb{R}$

all "expectations of functions"

Linearity of expectation

Theorem. *Let* X, Y *be two (numeric) random variables, and their joint distribution given by* $\mathbb{P}(X = x, Y = y)$*. The expectation is a "linear operator": that is,*

1. $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$,
2. $\mathrm{E}[cX] = c\mathrm{E}[X]$ for any fixed $c \in \mathbb{R}$.

$E[X+Y] = \sum (x+y) P(X=x, Y=y)$

$(X,Y) \to \mathbb{R}$ $f(X,Y) = X+Y$. See y'day slides for rest of proof

Some important distributions

Review: Bernoulli

$\Omega = \{0,1\}$ $\quad \mathbb{P}(X=0) = (1-p)$, $\mathbb{P}(X=1) = p$

$(\mathbb{P}(\{0\}))$

Bernoulli(p)

E.g. sending signal, $\mathbb{P}(\text{received}) = p$

Review: Binomial

n independent Bernoulli(p), denote $X_1, X_2, \ldots, X_n$

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i = k\right) = \sum_{\substack{x \in \Omega \\ \sum x_i = k}} \mathbb{P}(x) = \sum_{\substack{x \in \Omega \\ \sum x_i = k}} p^k (1-p)^{n-k} = \boxed{\binom{n}{k} p^k (1-p)^{n-k}}$$

$$\boxed{\text{Binomial}(n, p)}$$

Geometric

$$\boxed{P(X=k) = (1-p)^{k-1}p}$$

e.g. $\mathbb{P}(X=3) = (1-p)(1-p)p$

$f: \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ (not necessary)

$$\mathbb{P}(\Omega) = \sum_{i=1}^{\infty} (1-p)^{i-1}p = p\left(\sum_{i=1}^{\infty}(1-p)^{i-1}\right)$$

$$= p\left(\sum_{i=0}^{\infty}(1-p)^{i}\right)$$

$$= p\,\frac{1}{1-(1-p)} = p\left(\frac{1}{p}\right) = 1$$

Geometric(p)

$\mathbb{E}(\text{Geometric}(p))$?

Expectation equality

Let X be a random variable. If its sample space $\Omega \subset \mathbb{N}$, the following equality holds:

$$E[X] = \sum_{i=0}^{\infty} \mathbb{P}(X \geq i).$$

e.g. $\mathbb{P}(X=0) = 1$

$\mathbb{P}(\text{otherwise}) = 0$

$\sum i \mathbb{P}(i)$

$0 \cdot P(0)$

$P(1)$

$P(2)+P(2)$

$\boxed{\mathbb{P}(X \geq 0) = 1}$

$= 1 + 0 + 0 + \ldots + 0 = 0$

$E[X] = 0 \cdot 1 + 1 \cdot 0 + 2 \cdot 0 \ldots$

$= 0$

NOTE: this was all scratchwork to determine in lecture if it's $i=0$ or $i=1$ on the RHS

Expectation equality

$$E[X] = \sum_{i=1}^{\infty} i\,P(X=i)$$

$$= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} P(X=i)\right)$$

$$\overset{?}{=} \sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty} P(X=i)\right)$$

$$= \sum_{j=1}^{\infty} P(X \geq j)$$

① $\sum_{i=0}^{\infty} f(i)$ or $\sum_{i\in\mathbb{N}} f(i)$

② $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i,j)$ or $\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} f(i,j)$

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} f(i,j)$$

③ $\sum_{i=0}^{\infty}\sum_{j=0}^{i}$ or $\sum_{(i,j)\in A}$

$$\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}$$

$A = \{(i,j)\in\mathbb{N}\times\mathbb{N} \mid j \leq i\}$

To switch the order of a sum like $\sum_{i=1}^{\infty} \sum_{j=1}^{i} f(i,j)$:

1. Write $\sum_{i=1}^{\infty} \sum_{j=1}^{i} f(i,j) = \sum_{(i,j) \in A} f(i,j)$ for the appropriate index set $A \subset \mathbb{N} \times \mathbb{N}$. Here, it's $A = \{(i,j) \mid i \geq j\}$.

2. Put j term on outside, determine limits (here, it's $\sum_{j=1}^{\infty}$)

3. For each j, determine which i make the pair $(i,j) \in A$: here, it's $\sum_{i=j}^{\infty}$

4. Put it all together to get $\sum_{j=1}^{\infty} \sum_{i=j}^{\infty} f(i,j)$

Geometric revisited

$X = \text{Geom}(p)$

$$\mathbb{E}(X) = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i)$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1}$$

$$= \sum_{i=0}^{\infty} (1-p)^{i} = \frac{1}{1-(1-p)}$$

$$= \boxed{\frac{1}{p}}$$

eg. $p = 0.5$

$\mathbb{E}[X] = 2$

$p = 0.1$

$\mathbb{E}[X] = 10$

$p = 0.9$

$\mathbb{E}[X] = \frac{10}{9}$

where $\mathbb{P}(X \geq i) = (1-p)^{i-1}$ since this is exactly the event of failing the first $i-1$ times (we don't care what happens after)

Poisson

$$\boxed{\mathbb{P}(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}}$$

$\Omega = \{0,1,2,3,4,\dots\}$

$\lambda :=$ "Poisson parameter"

Poisson(λ)

$\boxed{\mathbb{E}[X] = \lambda}$ ✓

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Valid dist?

$$\mathbb{P}(X\in\Omega) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

Poisson

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \lambda e^{-\lambda} \left(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{k!} k \right)$$

$$= \lambda e^{-\lambda} \left(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right)$$

$$= \lambda e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) = \lambda e^{-\lambda} e^{\lambda} = \boxed{\lambda} \quad \blacksquare$$

Sum of Poisson

Suppose $X = \text{Poisson}(\lambda_1)$, $Y = \text{Poisson}(\lambda_2)$, and X, Y independent. Then $X+Y \sim \text{Poisson}(\lambda_1+\lambda_2)$.

(here, "$\sim$" means "has same distribution as")

Pf Writing out all possibilities where $X+Y=k$ yields the following:

$$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i) = \sum_{i=0}^{k} P(X=i)\,P(Y=k-i)$$

$$= \sum_{i=0}^{k} \frac{\lambda_1^i}{i!} e^{-\lambda_1} \frac{\lambda_2^{(k-i)}}{(k-i)!} e^{-\lambda_2}$$

(factoring out constant factors, & additional $k!$ term) $= e^{-\lambda_1-\lambda_2}\left(\frac{1}{k!}\right)\sum_{i=0}^{k} \frac{k!}{i!(k-i)!}\lambda_1^i \lambda_2^{k-i}$

(binomial theorem) $= \boxed{e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^k}{k!}}$

Poisson as limit of binomial

$\lim_{n\to\infty}$ Binomial$\left(n, \frac{\lambda}{n}\right) \sim$ Poisson(λ). Note intuitively: for $n=1$, this is akin to e.g. exactly at the 1 hr mark, checking if a call came or not. $n=2$ checks twice (every half hour), where in each interval it's now half as likely to get a call, and so on until you've divided up the hour infinitesimally small and get Poisson, which models "number of heads" in a continuous time interval.

Pf: See notes, well articulated there and would be messy here. Main identity needed is $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, which is proved in Rudin's Principles of Mathematical Analysis for those curious.